

# Optimal chattering solutions for longitudinal vibrations of a nonhomogeneous bar with clamped ends

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## Abstract

We consider a control problem for longitudinal vibrations of a nonhomogeneous bar with clamped ends. The vibrations of the bar are controlled by an external force which is distributed along the bar. For the minimization problem of mean square deviation of the bar we prove that the optimal control has an infinite number of switchings in a finite time interval, i.e., the optimal control is the chattering control.

## 1. Introduction

Consider small longitudinal vibrations of a nonhomogeneous bar of length  $l$ . The longitudinal displacement at a typical point  $x$  is denoted  $y(t, x)$  where  $t$  is the time. Let  $g(x, t)$  be a density of external longitudinal force at the instant of time  $t$  at the point  $x$ . Suppose that

$$g(t, x) = u(t)f(x)$$

where the force profile function  $f(x)$  is assumed to be given,  $u(t)$  is the control function. We assume that

$$-1 \leq u(t) \leq 1 \tag{1.1}$$

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The equation of longitudinal vibrations of the bar can be written as

$$p(x) y_{tt}(t, x) - (k(x) y_x(t, x))_x = u(t) f(x) \quad (1.2)$$

Here

$$p(x) = \rho(x) S(x), \quad k(x) = E(x) S(x)$$

where  $\rho(x)$  is the density of bar,  $S(x)$  is the cross-sectional area,  $E(x)$  is the Young's modulus at  $x$ , see, for example, [1, 2, 3].

We assume that the ends of the bar are clamped:

$$y|_{x=0} = y|_{x=l} = 0, \quad t > 0 \quad (1.3)$$

and the initial position and velocity are fixed:

$$y|_{t=0} = y_0(x), \quad x \in [0, l] \quad (1.4)$$

$$y_t|_{t=0} = y_1(x), \quad x \in [0, l] \quad (1.5)$$

We suppose that the coefficient functions  $k, p$  are smooth enough and

$$\forall x \in [0, l] \quad k(x) \geq k_0 > 0, \quad p(x) \geq p_0 > 0 \quad (1.6)$$

We consider an optimal control problem: to find such a control function  $u(t)$  that minimize the following functional

$$\int_0^\infty \int_0^l p(x) y^2(t, x) dx dt \rightarrow \inf \quad (1.7)$$

under (1.2)–(1.6).

The problems of longitudinal vibrations of a bar were considered in [1, 2, 3]. In [1, 2] the dynamics of the longitudinal vibrations of a bar subjected to viscous boundary conditions was studied. The optimal boundary control problem for longitudinal vibrations of a bar was considered in [3]. By using a maximum principle the optimal control was expressed in terms of an adjoint variable.

In this paper for the problem of controlling the longitudinal vibrations of a bar (1.1)–(1.7) we construct a solution  $y(t, x)$  in the form

$$y(t, x) = \sum_{j=1}^{\infty} s_j(t) h_j(x) \quad (1.8)$$

where  $\{h_j(x)\}_{j=1}^{\infty}$  are eigenfunctions of the Sturm-Liouville problem,  $\{s_j(t)\}_{j=1}^{\infty}$  are corresponding Fourier coefficients. To find Fourier coefficients we consider an optimal control problem in the space  $l^2$ . For the control problem in  $l^2$  we show that the optimal solutions contain singular trajectories and chattering trajectories. A trajectory is called a chattering trajectory if it has an infinite number of a control switchings on a finite time interval. By similar method we studied the optimal control problem for a rotating uniform Timoshenko beam [4, 5]. But for the Timoshenko beam similar results hold only for a dense set of initial conditions in the space  $l^2$ .

## 2. Optimal control problem in $l_2$

Define an operator  $L$  in  $C^2([0, l])$  by

$$Lh = (kh_x)_x$$

Consider the following Sturm-Liouville eigenvalue problem with Dirichlet boundary conditions:

$$Lh + \lambda p(x)h = 0, \quad x \in (0, l) \quad (2.1)$$

$$h(0) = 0, \quad h(l) = 0 \quad (2.2)$$

Here the functions  $k(x)$  and  $p(x)$  satisfy (1.6). It is known (see [6]) that the problem (2.1)–(2.2) has an infinite sequence of eigenvalues  $\{\lambda_j\}_{j=1}^{\infty}$ , which are simple and positive:

$$0 < \lambda_1 < \lambda_2 < \dots, \quad \lambda_j \rightarrow \infty, \quad j \rightarrow \infty$$

To each eigenvalue  $\lambda_j$  corresponds a single eigenfunction  $h_j$ , and the sequence of eigenfunctions  $\{h_j(x)\}_{j=1}^{\infty}$  forms an ortonormal basis of  $L_2((0, l); p)$  with the inner product  $(z, w)_p = \int_0^l p(x) z(x) w(x) dx$ . If  $k(x), p(x)$  are smooth enough (for example,  $k', p \in C^1([0, l])$ ) and the condition (1.6) holds, then the eigenvalues  $\lambda_j$  admit the asymptotic form [7, 8, 9]:

$$\frac{\lambda_j}{j^2} \sim \pi^2 \left( \int_0^l \sqrt{p(x)/k(x)} dx \right)^{-2}, \quad j \rightarrow \infty \quad (2.3)$$

Assume that  $y(t, \cdot) \in L_2((0, l); p)$ . For any  $t > 0$  we expand the solution  $y(t, x)$  of (1.2) in the basis  $\{h_j(x)\}_{j=1}^\infty$ :

$$\begin{aligned} y(t, x) &= \sum_{j=1}^{\infty} s_j(t) h_j(x) \\ s_j(t) &= \int_0^l p(x) y(t, x) h_j(x) dx = (y, h_j)_p \end{aligned} \quad (2.4)$$

Using (1.4)–(1.5) we get:

$$\begin{aligned} s_j(0) &= \int_0^l p(x) y_0(x) h_j(x) dx = (y_0, h_j)_p \\ \dot{s}_j(0) &= \int_0^l p(x) y_1(x) h_j(x) dx = (y_1, h_j)_p \end{aligned}$$

We multiply the equation (1.2) by  $h_j$  and integrate it in  $x$ :

$$\begin{aligned} \int_0^l (p y_{tt} + L y) h_j dx &= \int_0^l u f h_j dx \\ \frac{d^2}{dt^2} (y, h_j)_p + (L y, h_j) &= u (f, h_j) \Rightarrow \frac{d^2}{dt^2} (y, h_j)_p + (y, L h_j) = u (f, h_j) \end{aligned}$$

or

$$\frac{d^2}{dt^2} (y, h_j)_p + \lambda_j (y, h_j)_p = u (f, h_j)$$

Thus the function  $s_j(t)$  satisfies the following equation:

$$\ddot{s}_j(t) + \lambda_j s_j(t) = C_j u(t), \quad j = 1, 2, \dots$$

where

$$C_j = (f, h_j) = \int_0^l f(x) h_j(x) dx \quad (2.5)$$

We substitute (2.4) into (1.7). Using Parseval's equality we get:

$$\int_0^\infty \int_0^l p(x) y^2(t, x) dx dt = \int_0^\infty \sum_{j=1}^\infty s_j^2(t) dt \quad (2.6)$$

Denote  $\alpha_j = (y_0, h_j)_p$ ,  $\beta_j = (y_1, h_j)_p$ . We reduce the problem (1.2)–(1.7) to the following one:

$$\int_0^\infty \sum s_j^2(t) dt \rightarrow \inf \quad (2.7)$$

$$\ddot{s}_j(t) + \lambda_j s_j(t) = C_j u(t), \quad j = 1, 2, \dots \quad (2.8)$$

$$s_j(0) = \alpha_j, \quad \dot{s}_j(0) = \beta_j \quad (2.9)$$

$$-1 \leq u(t) \leq 1 \quad (2.10)$$

We shall assume everywhere below that

$$C_j \neq 0 \quad \text{for all } j = 1, 2, \dots \quad (2.11)$$

**Remark.** Assumption (2.11) is very essential for the problem (2.7)–(2.10). Indeed, let  $C_{j_0} = 0$  for some  $j_0$ . Then  $j_0$ -th equation in (2.8) takes the form

$$\ddot{s}_{j_0}(t) + \lambda_{j_0} s_{j_0}(t) = 0$$

Hence, if  $|\alpha_{j_0}| + |\beta_{j_0}| \neq 0$  then the corresponding solution  $s_{j_0}(t)$  does not vanish as  $t \rightarrow \infty$ . Therefore the integral (2.7) is equal to  $+\infty$  and the optimization problem (2.7)–(2.10) has not any sense.

Assume that

$$y_0, y_1 \in L_2((0, l); p), \quad f \in L_2(0, l) \quad (2.12)$$

Following [5] we denote

$$\omega_j = \sqrt{\lambda_j}, \quad \tau_j(t) = \dot{s}_j(t) / \omega_j, \quad c_j = C_j / \omega_j, \quad a_j = \alpha_j, \quad b_j = \beta_j / \omega_j$$

Then the problem (2.7)–(2.10) takes the form

$$\int_0^\infty \sum s_j^2(t) dt \rightarrow \inf \quad (2.13)$$

$$\dot{s}_j = \omega_j \tau_j, \quad \dot{\tau}_j = -\omega_j s_j + c_j u \quad (2.14)$$

$$s_j(0) = a_j, \quad \tau_j(0) = b_j, \quad j = 1, 2, \dots \quad (2.15)$$

$$-1 \leq u(t) \leq 1 \quad (2.16)$$

Denote

$$s(t) = (s_1(t), s_2(t), \dots), \quad \tau(t) = (\tau_1(t), \tau_2(t), \dots)$$

$$a = (a_1, a_2, \dots), \quad b = (b_1, b_2, \dots), \quad c = (c_1, c_2, \dots)$$

Consider the standart Hilbert space  $l_2$ :

$$l_2 = \left\{ w = (w_1, w_2, \dots) : w_n \in \mathbf{R}, \sum_{n=1}^{\infty} w_n^2 < \infty \right\}$$

with inner product  $(v, w) = \sum_{n=1}^{\infty} v_n w_n$ . Using assumption (2.12) we get that  $a, b, c \in l_2$ .

The existence and uniqueness of a solution  $(s(t), \tau(t))$  to problem (2.13)–(2.16) in the space  $l_2 \times l_2$  were proved in [5] for any initial data from an open neighborhood of the origin  $(s = 0, \tau = 0)$ . We apply a formal generalization of the Pontryagin maximum principle to the problem (2.13)–(2.16). Denote by  $\psi_i = (\psi_{i1}, \psi_{i2}, \dots)$  ( $i = 1, 2$ ) adjoint variables. Define the Pontryagin function

$$\begin{aligned} H(\psi_1, \psi_2, s, \tau, u) &= \sum_{j=1}^{\infty} (\psi_{1j} \omega_j \tau_j - \psi_{2j} \omega_j s_j + \psi_{2j} c_j u - s_j^2/2) = \\ &= H_0(\psi_1, \psi_2, s, \tau) + u H_1(\psi_1, \psi_2, s, \tau) \end{aligned}$$

where

$$H_0(\psi_1, \psi_2, s, \tau) = \sum_{j=1}^{\infty} \left( \psi_{1j} \omega_j \tau_j - \psi_{2j} \omega_j s_j - \frac{1}{2} s_j^2 \right), \quad H_1(\psi_1, \psi_2, s, \tau) = \sum_{j=1}^{\infty} \psi_{2j} c_j$$

For brevity we denote  $z = (\psi_1, \psi_2, s, \tau)$ . In the space  $l_2 \times l_2 \times l_2 \times l_2$  let us consider the Hamiltonian system

$$\begin{aligned} \dot{\psi}_{1j} &= \psi_{2j} \omega_j + s_j, & \dot{s}_j &= \omega_j \tau_j \\ \dot{\psi}_{2j} &= -\psi_{1j} \omega_j, & \dot{\tau}_j &= -\omega_j s_j + c_j u^*(t) \end{aligned} \quad j = 1, 2, \dots \quad (2.17)$$

where  $u^*(t)$  satisfies the following maximum condition:

$$u^*(t) = \arg \max_{u \in [-1, 1]} H(z(t), u) = \arg \max_{u \in [-1, 1]} (u H_1(z(t))) \quad (2.18)$$

Here we use notation:  $a^* = \arg \max_{a \in A} g(a)$  iff  $g(a^*) = \max_{a \in A} g(a)$ .

It was proved [5] that the Pontryagin maximum principle is the necessary and sufficient condition of optimality for the problem (2.13)–(2.16).

If  $H_1(z(t)) \neq 0$  along the trajectory the optimal control is uniquely determined as a function of time from the maximum condition (2.18):

$$u^*(t) = \text{sign}(H_1(z(t))) = \text{sign} \left( \sum_{j=1}^{\infty} \psi_{2j}(t) c_j \right)$$

Suppose that there exists an interval  $(t_1, t_2)$  such that

$$H_1(z(t)) \equiv 0, \quad \forall t \in (t_1, t_2)$$

To find an optimal control  $u(t)$  in this case we will differentiate the identity  $H_1(z(t)) \equiv 0$  by virtue of the system (2.17) until the control  $u$  with a non-zero coefficient occurs in the resulting expression with a non-zero coefficient.

$$\begin{aligned} \frac{d}{dt} \Big|_{(2.17)} H_1(z) &= \frac{d}{dt} \Big|_{(2.17)} \left( \sum_{j=1}^{\infty} \psi_{2j} c_j \right) = \left( - \sum_{j=1}^{\infty} c_j \psi_{1j} \omega_j \right) \\ \frac{d^2}{dt^2} \Big|_{(2.17)} H_1(z) &= \frac{d}{dt} \Big|_{(2.17)} \left( - \sum_{j=1}^{\infty} c_j \psi_{1j} \omega_j \right) = - \sum_{j=1}^{\infty} c_j (\psi_{2j} \omega_j^2 + s_j \omega_j) \\ \frac{d^3}{dt^3} \Big|_{(2.17)} H_1(z) &= - \frac{d}{dt} \Big|_{(2.17)} \sum_{j=1}^{\infty} c_j (\psi_{2j} \omega_j^2 + s_j \omega_j) = - \sum_{j=1}^{\infty} c_j \omega_j (-\omega_j^2 \psi_{1j} + \tau_j \omega_j) \\ \frac{d^4}{dt^4} \Big|_{(2.17)} H_1(z) &= \sum_{j=1}^{\infty} c_j \omega_j^2 (\psi_{2j} \omega_j^2 + 2s_j \omega_j) - u \sum_{j=1}^{\infty} c_j^2 \omega_j^2 \end{aligned} \quad (2.19)$$

Assume that all series in (2.19) are convergent in  $l^2$ . Denote

$$H_2(z) = - \sum_{j=1}^{\infty} c_j \psi_{1j} \omega_j, \quad H_3(z) = - \sum_{j=1}^{\infty} c_j \omega_j (\psi_{2j} \omega_j + s_j)$$

$$H_4(z) = - \sum_{j=1}^{\infty} c_j \omega_j^2 (-\psi_{1j} \omega_j + \tau_j)$$

From (2.19) it follows that

$$H_1(z(t)) = H_2(z(t)) = H_3(z(t)) = H_4(z(t)) = 0, \quad t \in (t_1, t_2)$$

We say a solution of the (2.17)–(2.18) is singular if it belongs to the surface

$$\Sigma = \{ z : H_1(z) = H_2(z) = H_3(z) = H_4(z) = 0 \} \quad (2.20)$$

A singular control  $u^0(t)$  is determined from the equation  $\frac{d^4}{dt^4} \Big|_{(2.17)} H_1(z) = 0$ .

Using (2.19) we obtain

$$u^0(t) = \sum_{j=1}^{\infty} c_j \omega_j^3 (\psi_{2j} \omega_j + 2s_j) / \sum_{j=1}^{\infty} c_j^2 \omega_j^2 \quad (2.21)$$

Note that the origin  $(\psi_1 = 0, \psi_2 = 0, s = 0, \tau = 0)$  is the singular trajectory and the corresponding singular control  $u^0(t)$  equals 0.

It was proved [5] that in a certain neighborhood of the origin the structure of the optimal solutions is the following one: for the finite time the optimal nonsingular trajectory enters the singular surface with infinite numbers of control switchings, after that the optimal trajectory remains on the singular surface and attains the origin for the infinite time. Namely, the following theorem holds.

*Theorem 1* [5]. Let  $c_j \neq 0 \quad \forall j$  and  $(c_1\omega_1^4, c_2\omega_2^4, c_3\omega_3^4, \dots) \in l_2$ . Assume that there exist positive constants  $\delta$  and  $K$  such that

$$|\omega_{j+1}| - |\omega_j| \geq \delta, \quad |\omega_j| \leq K \cdot j, \quad j = 1, 2, \dots$$

Then there exists an open neighborhood of the origin in the space  $(s, \tau)$  such that the following statements hold for all initial data  $(a, b)$  from this neighborhood.

- (i) The problem (2.13)–(2.16) has a unique optimal solution.
- (ii) In the space  $z = (\psi_1, \psi_2, s, \tau)$  there exists the singular surface  $\Sigma$  of codimension 4 given by the equations

$$\begin{aligned} \sum_{j=1}^{\infty} \psi_{2j} c_j &= 0, & \sum_{j=1}^{\infty} c_j \psi_{1j} \omega_j &= 0 \\ \sum_{j=1}^{\infty} c_j \omega_j (\psi_{2j} \omega_j + s_j) &= 0, & \sum_{j=1}^{\infty} c_j \omega_j^2 (-\psi_{1j} \omega_j + \tau_j) &= 0 \end{aligned}$$

which is filled in by the singular extremals of the problem (2.13)–(2.16). The control on singular extremals are defined by (2.21).

- (iii) For all initial data not belonging to the projection of the singular surface  $\Sigma$  on the space  $(s, \tau)$ , the optimal trajectories arrive at  $\Sigma$  in finite time with countable many control switchings, i.e., the optimal trajectories are chattering trajectories.



### 3. Optimal solution for controlling vibrations

Let  $(s^*(t), u^*(t))$  be an optimal solution of (2.13)–(2.16). Consider

$$y^*(t, x) = \sum_{j=1}^{\infty} s_j^*(t) h_j(x) \quad (3.1)$$

where  $\{h_j(x)\}_{j=1}^{\infty}$  are eigenfunctions of the Sturm-Liouville problem (2.1)–(2.2).

Series (3.1) formally satisfies equation (1.2), boundary conditions (1.3) and initial conditions (1.4)–(1.5). We will show that this series gives a weak solution of problem (1.2)–(1.5).

Denote  $Q_T = (0, l) \times (0, T)$ , where  $T > 0$ . Consider the Sobolev space  $H^k(Q_T) = W_2^k(Q_T)$ ,  $k \geq 0$ . The space  $H^k(Q_T)$  consists of all functions  $v \in L_2(Q_T)$  whose generalized derivatives up to order  $k$  exist and belong to  $L_2(Q_T)$ . The space  $H_0^k(Q_T)$  can be defined as a completion of  $C_0^\infty(Q_T)$  with respect to the norm of the space  $H^k(Q_T)$ .

Let  $g = uf \in L_2(Q_T)$ ,  $y_1 \in L_2(0, l)$ .

*Definition 1.* We say a function  $y \in H^1(Q_T)$  is a weak solution of the (1.2)–(1.5) if

$$(i) \quad y|_{x=0} = y|_{x=l} = 0, \quad t > 0, \quad y|_{t=0} = y_0(x), \quad x \in [0, l];$$

(ii)

$$\int_{Q_T} (ky_x v_x - py_t v_t) dx dt = \int_{Q_T} g v dx dt + \int_0^l y_1(x) v(0, x) dx$$

for each  $v \in H^1(Q_T)$ :  $v|_{x=0} = v|_{x=l} = 0$ ,  $v|_{t=T} = 0$ .

*Definition 2.* We say  $y \in H^2(Q_T)$  is a almost everywhere solution of the problem (1.2)–(1.5) provided  $y$  satisfies equation (1.2) in  $Q_T$  for almost all  $(t, x) \in Q_T$  and  $y$  satisfies (1.3)–(1.5).

The following theorem is the main result for the problem (1.2)–(1.7).

*Theorem 2.* Let  $y_0 \in H_0^2(0, l)$ ,  $y_1 \in H_0^1(0, l)$ ,  $k(x), p(x)$  are smooth enough (for example,  $k, p \in C^4([0, l])$ ),  $p(x) \geq p_0 > 0$ ,  $k(x) \geq k_0 > 0$ . Assume that  $f \in C^4[0, l]$ ,

$$f(0) = f(l) = 0, \quad f^{(i)}(0) = f^{(i)}(l) = 0, \quad i = 1, 2, 3 \quad (3.2)$$

and condition (2.11) holds. Then there exist positive constants  $q_1$  and  $q_2$  such that if

$$\|y_0\|_{L_2((0,l);p)} < q_1, \quad \|y_1\|_{L_2((0,l);p)} < q_2$$

then

- (i) the problem (1.2)–(1.7) has a unique optimal solution  $y^*(t, x)$ ;
- (ii)  $y^* \in H^2(Q_T)$  for all  $T > 0$ ;
- (iii) an optimal solution  $y^*(t, x)$  has an infinite number of control switchings in a finite time interval.

*Proof.* Here we use notations introduced in Section 2. Since the functions  $p, k, f$  satisfy conditions of Theorem 2 it follows [8] that  $C_j \sim j^{-4}$  as  $j \rightarrow \infty$ , where

$$C_j = (f, h_j) = \int_0^l f(x) h_j(x) dx$$

Then we have

$$\sum_{j=1}^{\infty} (c_j \omega_j^4)^2 = \sum_{j=1}^{\infty} (C_j \omega_j^3)^2 = \sum_{j=1}^{\infty} C_j^2 \lambda_j^3 < \infty \implies (c_1 \omega_1^4, c_2 \omega_2^4, c_3 \omega_3^4, \dots) \in l_2$$

The property (2.3) of the eigenvalues  $\{\lambda_j\}_{j=1}^{\infty}$  of the problem (2.1) – (2.2) imply that there exist positive constants  $\delta$  and  $B$  such that

$$|\omega_{j+1}| - |\omega_j| \geq \delta, \quad |\omega_j| \leq Bj$$

Now we may apply Theorem 1 to the problem (2.13)–(2.16). We get that the optimal control  $u^*(t)$  for the problem (2.13)–(2.16) has an infinite number of switchings in the finite time interval.

Since the functions  $f, \alpha, \beta$  satisfy the conditions of Theorem 2 it follows (see [10, 11]) that the function  $y^*(t, x)$  defined by (3.1) is a unique weak solution of the problem (1.2)–(1.5) and  $y^* \in H^2(Q_T)$ . Hence [10]  $y^*(t, x)$  is the solution almost everywhere of the problem (1.2)–(1.5). Thus the function  $y^*(t, x)$  satisfies (1.2) for almost all  $(t, x) \in (0, l) \times (0, +\infty)$ , boundary and initial conditions (1.3)–(1.5).

Since the function  $s^*(t) = (s_1^*(t), s_2^*(t), \dots)$  minimizes the functional (2.7) and the identity (2.6) holds then the function  $y^*(t, x)$  minimizes the functional (1.7). Thus  $y^*(t, x)$  is a solution of the problem (1.1)–(1.7).

#### 4. Conclusion

We considered the optimal control problem of longitudinal vibrations of a nonhomogeneous bar with clamped ends. We proved that the optimal trajectories contain singular part and nonsingular one with accumulation of control switchings.

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